Calculation of the differential cross section for Compton scattering in tree-level quantum electrodynamics

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Abstract

The differential cross section for unpolarised Compton scattering is calculated with respect to both t and $\cos\theta$. $|\mathcal{M}(e^-\gamma \to e^-\gamma)|^2$ is evaluated to first order in α_{em} using perturbative QED. The Klein-Nishina formula is derived using two different approaches, working both from the centre of mass and lab frames. Plots of $d\sigma/dt$ and $d\sigma/d\cos\theta$ resemble classical Thomson scattering at low $\sqrt{s}$. At higher energy $d\sigma/d\cos\theta$ is largest at $\theta = 0$, decreasing rapidly towards π. The overall cross section decreases as $\sqrt{s}$ increases.

1 Introduction

Compton scattering occurs when electromagnetic radiation is scattered by free electrons at rest in the lab reference frame. The initial and final states are an electron and a photon: $e^-\gamma \to e^-\gamma$. The cross section of this interaction is intrinsic to the colliding particles and allows us to calculate the probability of this final state, independently of the luminosity L of any particular experiment. The number of corresponding scattering events N is related to the cross section by

$$N = L\sigma. \tag{1.1}$$

The general formula for the infinitesimal cross section of a two-particle collision is given by [3]

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} |\mathcal{M}(p_A, p_B \to \{p_f\})|^2 d\Pi_n, \tag{1.2}$$

where the phase space integral over the final states has the form:

$$\int d\Pi_n = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f). \tag{1.3}$$

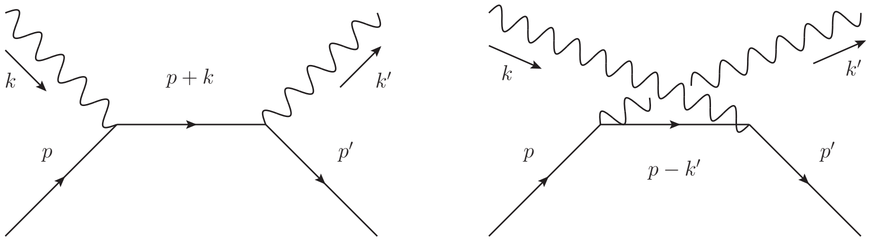


Figure 2.1: Feynman diagrams for Compton scattering. Time runs left to right.

The last two factors in (1.2) are Lorentz invariant, while the first is invariant under co-linear boosts. The square matrix element $|\mathcal{M}|^2$ may be calculated using the Feynman rules of quantum electrodynamics (QED) without any reference to a particular frame of reference. We will undertake this calculation first in Section 2, before returning, in Section 3, to evaluate the remaining factors in both the centre of mass and lab frames. This will allows us to calculate the differential cross section with respect to both: the scattering angle in the lab frame $d\sigma/d\cos\theta$, and the square momentum transfer between initial-state and final-state photons $d\sigma/dt$. In Section 4 we plot $d\sigma/dt$ and $d\sigma/d\cos\theta$ for three different of centre of mass energies $\sqrt{s}$. Conclusions are made in Section 5.

2 Calculating the square S-matrix element

2.1 Applying the Feynman rules

At zero loop (tree) level, there are two possible Feynman diagrams with initial and final states representative of Compton scattering. These are shown in figure 2.1.

Following the reverse fermion flow and applying the QED Feynman rules [1] to each diagram, we may immediately write down the corresponding transition amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$,[1]

$$i\mathcal{M}_1 = \bar{u}^{s'}(p')(-ie\gamma^\mu)\epsilon^{*r'}_\mu(k')\left(\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\right)(-ie\gamma^\nu)\epsilon^r_\nu(k)u^s(p) \tag{2.1}$$

$$i\mathcal{M}_2 = \bar{u}^{s'}(p')(-ie\gamma^\nu)\epsilon^r_\nu(k)\left(\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\right)(-ie\gamma^\mu)\epsilon^{*r}_\mu(k')u^s(p) \tag{2.2}$$

In the canonical quantisation of QFT these amplitudes are terms in the Wick expansion of the scattering matrix element $\langle f|\, S\, |i\rangle$ [4]. Therefore, the total transition amplitude is equal to the sum of these expressions,

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \tag{2.3}$$

In accordance with Fermi-Dirac statistics, there is no relative minus sign between the two terms due to the identical fermion flow in both diagrams. In addition,

[1]Note that spin notation is coupled to momentum notation and so spin superscripts are henceforth left implicit. In addition, we will implement the notation $u = u(p)$, $u' = u(p')$, $\epsilon = \epsilon(k)$ and $\epsilon' = \epsilon(k')$.

we are uninterested in terms corresponding to processes where no scattering occurs and so we have

$$\langle f|\, S - 1\, |i\rangle = i\mathcal{M}(2\pi)^4 \delta^{(4)}(P_F - P_I). \tag{2.4}$$

The momentum conserving delta function is shared by all S-matrix elements, thus it is absorbed in to the general formula for a cross section (1.2). In order to calculate this formula we require $|\mathcal{M}|^2$. This is given by[2]

$$|\mathcal{M}|^2 = (\mathcal{M}_1 + \mathcal{M}_2)(\mathcal{M}_1^* + \mathcal{M}_2^*) = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\mathcal{R}(\mathcal{M}_1\mathcal{M}_2^*). \tag{2.5}$$

The total square matrix element is therefore equal to the sum of three terms which may be calculated separately: two terms simply equal to the square of each amplitude and a third 'interference' term between the diagrams. We shall begin by evaluating $|\mathcal{M}_1|^2$ in full detail before returning to the later two terms.

2.2 Calculating $|\mathcal{M}_1|^2$

We may simplify the expression for $\mathcal{M}_1$ (2.1). First, by rearranging commuting terms,

$$i\mathcal{M}_1 = -ie^2 \epsilon'^*_\mu \epsilon_\nu \bar{u}' \left(\frac{\gamma^\mu (\not{p} + \not{k} + m)\gamma^\nu}{(p+k)^2 - m^2} \right) u. \tag{2.6}$$

The denominator may then be rewritten,

$$\begin{aligned} (p+k)^2 - m^2 &= p^2 + p\cdot k + k \cdot p + k^2 - m^2 \\ &= 2p\cdot k, \end{aligned} \tag{2.7}$$

where the second equality follows from $p^2 = m^2$ and $k^2 = 0$. The numerator may also be simplified using

$$\begin{aligned} (\not{p} + m)\gamma^\nu u &= (\gamma^\mu \gamma^\nu p_\mu + \gamma^\nu m)u \\ &= (2p^\nu - \gamma^\nu \not{p} + \gamma^\nu m)u \\ &= 2p^\nu u, \end{aligned} \tag{2.8}$$

where the second equality follows from the Dirac algebra $[\gamma^\mu, \gamma^\nu] = 2\eta^{\mu\nu}$ and the third from the equation of motion for positive frequency solutions of the free Dirac field $(\not{p} - m)u = 0$. With $\mathcal{M}_1$ simplified,

$$i\mathcal{M}_1 = -ie^2 \epsilon'^*_\mu \epsilon_\nu \bar{u}' \left(\frac{2\gamma^\mu p^\nu + \gamma^\mu \not{k} \gamma^\nu}{2p\cdot k} \right) u, \tag{2.9}$$

we may find the complex conjugate $\mathcal{M}_1^*$,[3]

$$-i\mathcal{M}_1^* = +ie^2 \epsilon'_\rho \epsilon^*_\sigma \bar{u} \left(\frac{2\gamma^\rho p^\sigma + \gamma^\sigma \not{k} \gamma^\rho}{2p\cdot k} \right) u'. \tag{2.10}$$

[2] As the matrix elements are commutative, the second equality follows simply from

$$\mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_1^*\mathcal{M}_2 = (a+ib)(c-id) + (c+id)(a-ib) = 2(ac+bd) = 2\mathcal{R}(\mathcal{M}_1\mathcal{M}_2^*).$$

Hence, the square matrix element is

$$|\mathcal{M}_1|^2 = e^4 \epsilon'^*_\mu \epsilon'_\rho \epsilon_\nu \epsilon^*_\sigma \bar{u}' \left(\frac{2\gamma^\mu p^\nu + \gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} \right) u\bar{u} \left(\frac{2\gamma^\sigma p^\rho + \gamma^\sigma \not{k} \gamma^\rho}{2p \cdot k} \right) u'. \tag{2.11}$$

If we make the spinor indices explicit we are free to rearrange this,

$$\begin{aligned} |\mathcal{M}_1|^2 &= e^4 \epsilon'^*_\mu \epsilon'_\rho \epsilon_\nu \epsilon^*_\sigma \bar{u}'_a \left(\frac{2\gamma^\mu p + \gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} \right)_{ab} u_b \bar{u}_c \left(\frac{2\gamma^\sigma p^\rho + \gamma^\sigma \not{k} \gamma^\rho}{2p \cdot k} \right)_{cd} u'_d \\ &= e^4 \epsilon'^*_\mu \epsilon'_\mu \epsilon_\nu \epsilon^*_\nu u'_d \bar{u}'_a \left(\frac{2\gamma^\mu p + \gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} \right)_{ab} u_b \bar{u}_c \left(\frac{2\gamma^\sigma p^\rho + \gamma^\sigma \not{k} \gamma^\rho}{2p \cdot k} \right)_{cd}. \end{aligned} \tag{2.12}$$

2.2.1 Summing over polarisation states

Currently the expression retains freely specified spin and polarization states for the electrons and photons. Experimental coulomb scattering, however, typically involves unpolarised photons colliding with electrons in an unpolarised medium and so we must average over these states. Likewise, photon and electron detectors are also commonly blind to polarisation and so we must sum over the final spin states. Therefore, we are interested in the unpolarised square matrix element

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{s'} \sum_{r'} |\mathcal{M}(s, r \to s', r')|^2 = \frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \overline{|\mathcal{M}|^2}. \tag{2.13}$$

The spin summation for the electron states can be performed using the completeness relation

$$\sum_{s=1,2} = u^s(p)\bar{u}^s(p) = \not{p} + m. \tag{2.14}$$

While the polarisation summation for the photon polarizations can be performed by making the replacement

$$\sum_{r=1,2} = \epsilon^{*r}_\mu(k)\epsilon^r_\nu(k) \to -g_{\mu\nu}, \tag{2.15}$$

which is valid here as both Lorentz indices are contracted within the expression for $\mathcal{M}$. Summing over spin states in (2.12) by using these techniques we arrive at the expression

$$\begin{aligned} \overline{|\mathcal{M}_1|^2} = \frac{e^4}{4} g_{\mu\rho} g_{\nu\sigma} (\not{p}' + m)_{da} \left(\frac{2\gamma^\mu p^\nu + \gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} \right)_{ab} \\ (\not{p} + m)_{bc} \left(\frac{2\gamma^\sigma p^\rho + \gamma^\sigma \not{k} \gamma^\rho}{2p \cdot k} \right)_{cd}. \end{aligned} \tag{2.16}$$

[3] The complex conjugate of a bi-spinor product $\bar{v}\gamma^\mu u$ may be calculated,

$$\begin{aligned} (\bar{v}\gamma^\mu u)^* &= u^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger v = u^\dagger (\gamma^\mu)^\dagger \gamma^0 v = u^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 v = (u^\dagger \gamma^0)\gamma^\mu v \\ &= \bar{u}\gamma^\mu v. \end{aligned}$$

2.2.2 Trace algebra

An examination of (2.16) reveals that it has the form of a trace of a product of four 4-dimensional matrices,

$$\mathrm{tr}(MNOP) = \sum_{d=0}^{3} (mnop)_{dd} = \sum_{c,b,a,d=0}^{3} m_{da} n_{ab} o_{bc} p_{cd} \tag{2.17}$$

Therefore, we have:

$$\overline{|\mathcal{M}_1|^2} = \frac{e^4}{4} \frac{1}{(2p \cdot k)^2} \mathcal{A}, \tag{2.18}$$

where,

$$\mathcal{A} = \mathrm{tr}[(\not{p}' + m)(\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\nu \not{k} \gamma_\mu + 2\gamma_\mu p_\nu)] \tag{2.19}$$

On expansion $\mathcal{A}$ consists of $2^4 = 16$ terms,

$$\mathcal{A} = \mathcal{I} + \mathcal{II} + \cdots + \mathcal{XVI}. \tag{2.20}$$

These traces may be evaluated individually by hand using the suitable contraction identities [3]. If, in multiplying out, we move from the end of the expression to the start while cycling within brackets from left to right, we find that terms $\mathcal{III}$, $\mathcal{IV}$, $\mathcal{VII} - \mathcal{X}$, $\mathcal{XIII}$ and $\mathcal{XIV}$ contain an odd number of gamma matrices and, consequently, immediately vanish. Appendix A contains the evaluations of the other eight traces. Adding all these terms together returns

$$\mathcal{A} = 32[2m^4 + 2m^2(p \cdot k) - m^2(p \cdot p') - m^2(p' \cdot k) + (p \cdot k)(p' \cdot k)]. \tag{2.21}$$

2.2.3 Mandelstam variables

It is helpful at this point to utilise the Lorentz invariant Mandelstam variables, which follow directly from 4-momentum conservation $p + k = p' + k'$,

$$\begin{aligned} s &= (p + k)^2 = 2p \cdot k + m^2 \\ &= (p' + k')^2 = -2p' \cdot k' + m^2, \\ t &= (p' - p)^2 = -2p \cdot p' + 2m^2 \\ &= (k' - k)^2 = -2k \cdot k', \\ u &= (p' - k)^2 = -2k \cdot p' + m^2 \\ &= (k' - p)^2 = -2p \cdot k' + m^2. \end{aligned} \tag{2.22}$$

These also yield the relation $s + t + u = 2m^2$. Substituting these in to $\mathcal{A}$ gives:

$$\begin{aligned} \mathcal{A} &= 8[2m^2(2s + t + u) + (s - m^2)(u - m^2)] \\ &= 8[4m^4 + 2m^2(s - m^2) - (s - m^2)(u - m^2)]. \end{aligned} \tag{2.23}$$

Thus,

$$\overline{|\mathcal{M}_1|^2} = 2e^4 \left[\frac{4m^4}{(s - m^2)^2} + \frac{2m^2}{(s - m^2)} - \frac{(u - m^2)}{(s - m^2)} \right]. \tag{2.24}$$

Now we may return to calculate the other terms in $|\mathcal{M}|^2$.

2.3 Calculating the full square matrix element

We may simplify $\mathcal{M}_2$ in the same way as $\mathcal{M}_1$,

$$i\mathcal{M}_2 = -ie^2\epsilon'^*_\nu\epsilon_\mu\bar{u}'\left(\frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right)u, \tag{2.25}$$

and, similarly, find its complex conjugate,

$$-i\mathcal{M}_2^* = +ie^2\epsilon'_\rho\epsilon^*_\sigma\bar{u}\left(\frac{-\gamma^\rho\not{k}'\gamma^\sigma + 2\gamma^\sigma p^\rho}{-2p\cdot k'}\right)u'. \tag{2.26}$$

With these in hand we may find

$$|\mathcal{M}_2|^2 = e^4\epsilon'^*_\mu\epsilon'_\rho\epsilon_\nu\epsilon^*_\sigma\bar{u}'\left(\frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right)u \\ \bar{u}\left(\frac{-\gamma^\rho\not{k}'\gamma^\sigma + 2\gamma^\sigma p^\rho}{-2p\cdot k'}\right)u'. \tag{2.27}$$

Comparing this with (2.11) we see that $|\mathcal{M}_1|^2$ is identical to $|\mathcal{M}_2|^2$ under the replacement $k \to -k'$ ($s \to u$). This means that the evaluation is symmetric and immediately allows us to write,

$$\overline{|\mathcal{M}_2|^2} = 2e^4\left[\frac{4m^4}{(u-m^2)^2} + \frac{2m^2}{(u-m^2)} - \frac{(s-m^2)}{(u-m^2)}\right]. \tag{2.28}$$

Now all that is left is to calculate the interference term, which from (2.9) and (2.26) is

$$2\mathcal{M}_1\mathcal{M}_2^* = 2e^4\epsilon'^*_\mu\epsilon'_\rho\epsilon_\nu\epsilon^*_\sigma\bar{u}'\left(\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k}\right)u \\ \bar{u}\left(\frac{-\gamma^\sigma\not{k}'\gamma^\rho + 2\gamma^\sigma p^\rho}{-2p\cdot k'}\right)u'. \tag{2.29}$$

Due to the symmetry between this expression and (2.11) we see that its evaluation will mirror Sections 2.2.1 and 2.2.2 with expressions equivalent to (2.18) and (2.19) of

$$\overline{2\mathcal{M}_1\mathcal{M}_2^*} = \frac{e^4}{2}\frac{1}{(2p\cdot k)(-2p\cdot k')}\mathcal{A}', \tag{2.30}$$

where,

$$\mathcal{A}' = \mathrm{tr}[(\not{p}' + m)\left(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu\right)(\not{p} + m)\left(-\gamma_\mu\not{k}'\gamma_\nu + 2\gamma_\nu p_\mu\right)]. \tag{2.31}$$

When expanded out $\mathcal{A}'$ likewise comprises 16 terms,

$$\mathcal{A}' = \mathcal{I}' + \mathcal{II}' + \cdots + \mathcal{XVI}'. \tag{2.32}$$

Multiplying out identically to (2.21), the equivalent terms immediately vanish. Appendix A contains the evaluations of the other 8 traces. Adding all these terms together,

$$\begin{aligned}\mathcal{A}' =&32(k\cdot k')(p\cdot p') - 32(p\cdot k)(p\cdot p')\\ &+16m^2(p'\cdot k) + 32(p\cdot k')(p\cdot p')-\\ &-16m^2(p'\cdot k') + 16m^2(p\cdot p') - 16m^2(k\cdot k')\\ &+32m^2(p\cdot k) - 32(p\cdot k') + 16m^4.\end{aligned}$$

Substituting the Mandelstam variables gives us:

$$\mathcal{A}' = 8\left[4m^4 + m^2(u-m^2) + m^2(s-m^2)\right]. \tag{2.33}$$

Therefore,

$$\overline{2\mathcal{M}_1\mathcal{M}_2^*} = 2e^2\left[\frac{8m^4}{(s-m^2)(u-m^2)} + \frac{2m^2}{(u-m^2)} - \frac{2m^2}{(s-m^2)}\right]. \tag{2.34}$$

Bringing all the terms together finally gives us:

$$\begin{aligned}\overline{|\mathcal{M}|^2} = 2e^4\Bigg[&4m^4\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)^2\\ &+4m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right) - \frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2}\Bigg].\end{aligned} \tag{2.35}$$

To calculate the cross section from this we must now specify the frame of reference.

3 Frame-specific evaluation

3.1 Centre of mass frame

In the centre of mass (CM) frame the 4-momenta of the particles may be written:

$$\begin{aligned} k =&(\omega^*, 0, 0, \omega^*), \qquad & k' &= (\omega^*, \omega^*\sin\theta^*, 0, \omega^*\cos\theta^*),\\ p =&(E, 0, 0, -\omega^*), \qquad & p' &= (E, -\omega^*\sin\theta^*, 0, -\omega^*\cos\theta^*),\end{aligned} \tag{3.1}$$

where ω^* and θ^* are the photon energy and scattering angle in the CM frame. This makes the Mandelstam variables:

$$\begin{aligned} s &= 2\omega^*(E+\omega^*) + m^2,\\ t &= 2\omega^{*2}(\cos\theta^* - 1),\\ u &= -2\omega^*(E+\omega^*\cos\theta^*) + m^2.\end{aligned} \tag{3.2}$$

Therefore, (1.2) becomes:

$$d\sigma = \frac{1}{2E2\omega^*|1-\frac{\omega^*}{E}|}\overline{|\mathcal{M}|^2}d\Pi_2. \tag{3.3}$$

In terms of frame invariant variables,

$$d\sigma = \frac{1}{2(s-m^2)}\overline{|\mathcal{M}|^2}d\Pi_2. \tag{3.4}$$

3.2 Evaluating the phase space integral in the CM frame

We may also evaluate the final state phase space integral in the CM frame,[4]

$$\int d\Pi_2 = \iiint \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}} \iiint \frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{k'}}(2\pi)^4\delta^{(4)}(p+k-p'-k'). \quad (3.5)$$

We integrate over all 3 components of $\mathbf{p'}$ which, given $\mathbf{p} = -\mathbf{k}$, sets $\mathbf{p'} = -\mathbf{k'}$. And so with a change of variable to spherical polar coordinates in phase space we have,[5]

$$\int d\Pi_2 = \iiint \frac{dk'k'^2 d\cos\theta^* d\phi^*}{(2\pi)^2 4E_{p'}E_{k'}}\delta(E+\omega^* - E_{p'}(k') - E_{k'}(k')). \quad (3.6)$$

However, the collision is symmetric about ϕ so we pick up another factor of 2π. We now have an integral over momentum, but a Dirac delta function in terms of energies, which are functions of momentum via the dispersion relation. So, if $f(k') = E + \omega^* - E_{p'}(k') - E_{k'}(k')$, we must evaluate

$$\begin{aligned}\int d\Pi_2 &= \iint \frac{dk'k'^2 d\cos\theta^*}{8\pi E_{p'}E_{k'}}\frac{\delta(k'-k'_0)}{\left|\frac{\partial f(k'_0)}{\partial k'}\right|} \\ &= \int \frac{k'^2 d\cos\theta^*}{8\pi E_{p'}E_{k'}}\left(\frac{k'}{E_{p'}} + \frac{k'}{E_{k'}}\right)^{-1} \\ &= \int \frac{k'^2 d\cos\theta^*}{8\pi}\left(\frac{k'}{E_{p'}+E_{k'}}\right), \end{aligned} \quad (3.7)$$

where $f(k'_0) = 0$. The exact value of k'_0 is unnecessary, but integrating over this distribution enforces energy conservation $E + w^* = E_{p'} + E_{k'}$. Hence,

$$\int d\Pi_2 = \int d\cos\theta^* \frac{\omega^*}{8\pi(E+\omega^*)}. \quad (3.8)$$

We may also calculate that in the CM frame $dt = 2w^{*2} d\cos\theta^*$. Thus,

$$\begin{aligned}\int d\Pi_2 &= \int \frac{dt}{16\pi\omega^*(E+\omega^*)} \\ &= \int \frac{dt}{8\pi(s-m^2)}. \end{aligned} \quad (3.9)$$

This gives us a manifestly Lorentz invariant formula for the cross section,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s-m^2)^2}\overline{|\mathcal{M}|^2}, \quad (3.10)$$

where $\overline{|\mathcal{M}|^2}$ is given by (2.35). To express this in terms of more physical variables we now move in to the lab frame.

[4]We temporarily forget manual conservation of energy and momentum in order for the Dirac delta function to take over.

[5]When working in spherical polar coordinates k' is one dimensional momentum, not k'^μ.

3.3 Lab frame

In the lab frame the electron is at rest and the 4-momenta of the particles may be written:

$$\begin{aligned} k =&(\omega, 0, 0, \omega) & k' &= (\omega', \omega' \sin\theta, 0, \omega' \cos\theta) \\ p =&(m, \mathbf{0}) & p' &= (E, -\mathbf{p}') \end{aligned} \tag{3.11}$$

This makes the Mandelstam variables in the lab frame:

$$\begin{aligned} s &= 2mw + m^2, \\ t &= 2ww'(\cos\theta - 1), \\ u &= -2mw' + m^2. \end{aligned} \tag{3.12}$$

Note that in this frame w' and θ are not independent variables. To find this dependence we derive the Compton formula. Conserving 4-momentum $p + k = p' + k'$ again,

$$\begin{aligned} p'^2 &= (p + k - k')^2, \\ &= p^2 + 2p \cdot (k - k') - 2k \cdot k' + k^2 + k'^2, \\ m^2 &= m^2 + 2m(w - w') + 2ww'(\cos\theta - 1). \end{aligned} \tag{3.13}$$

Thus,

$$\frac{1}{w'} - \frac{1}{w} = \frac{1}{m}(1 - \cos\theta). \tag{3.14}$$

Rearranging for w',

$$w' = \frac{w}{1 + \frac{w}{m}(1 - \cos\theta)}, \tag{3.15}$$

which allows us to express t only in terms of $\cos\theta$,

$$t = \frac{2w^2(1 - \cos\theta)}{1 + \frac{w}{m}(1 - \cos\theta)} \tag{3.16}$$

$$\frac{dt}{d\cos\theta} = \frac{2w^2}{\left(1 + \frac{w}{m}(1 - \cos\theta)\right)^2} = 2w'^2. \tag{3.17}$$

So, we find the differential cross section with respect to the angle in the lab frame,

$$\frac{d\sigma}{d\cos\theta} = \frac{d\sigma}{dt}\frac{dt}{d\cos\theta} = 2w'^2 \frac{1}{16\pi(2mw)^2}\overline{|\mathcal{M}|^2}.$$

To express $\overline{|\mathcal{M}|^2}$ in terms of lab frame variables we substitute (3.12) and (3.15) in to (2.35) to get

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= 2e^4 \left[4m^4 \left(\frac{\cos\theta - 1}{2m^2}\right)^2 + 4m^2 \left(\frac{\cos\theta - 1}{2m^2}\right) + \frac{w'}{w} + \frac{w}{w'}\right] \\ &= 2e^4 \left[\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta\right], \end{aligned} \tag{3.18}$$

which finally produces the Klein-Nishina formula [2],

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{w'}{w}\right)^2 \left[\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta\right]. \tag{3.19}$$

3.4 Evaluating the phase space integral in the lab frame

In order to verify this result it is possible to return to the phase space integral and carry it out directly in the rest frame of the incident electron,

$$\int d\Pi_2 = \iiint \frac{d^3p'}{(2\pi)^3}\frac{1}{2w'} \iiint \frac{d^3k'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(p+k-p'-k'). \quad (3.20)$$

Integrating once again over all three components of **p'** sets $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$ and leaves the remaining delta function over the energies. Changing to spherical polar coordinates again,

$$\int d\Pi_2 = \iint \frac{d\omega'\omega'^2 d\cos\theta}{8\pi E'\omega'}\delta(g(\omega')), \quad (3.21)$$

where,

$$\begin{aligned} g(\omega') &= E' + \omega' - m - \omega \\ &= \sqrt{(k-k')^2 + m^2} + \omega' - m - \omega \\ &= \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta} + \omega' - m - \omega \quad (3.22) \\ \left|\frac{\partial g(\omega')}{\partial \omega'}\right| &= 1 + 2(\omega' - \omega\cos\theta)\frac{1}{2}(m^2 + \omega + \omega'^2 - 2\omega\omega'\cos\theta)^{-\frac{1}{2}} \\ &= 1 + \frac{\omega' - \omega\cos\theta}{E'}. \quad (3.23) \end{aligned}$$

Equation (3.20) then becomes,

$$\begin{aligned} \int d\Pi_2 &= \iint \frac{d\omega'\omega' d\cos\theta}{8\pi E'}\left(1 + \frac{\omega' - \omega\cos\theta}{E'}\right)^{-1}\delta(\omega' - \omega'_0) \\ &= \int \frac{d\cos\theta}{8\pi}\frac{\omega'}{E' + \omega' - \omega\cos\theta} \\ &= \int \frac{d\cos\theta}{8\pi}\frac{\omega'}{m\left(1 + \frac{\omega}{m}(1-\cos\theta)\right)} \\ &= \int \frac{d\cos\theta}{8\pi}\frac{\omega'^2}{m\omega}, \quad (3.24) \end{aligned}$$

where for the 3rd equality, integrating over the delta function enforces that $E' + \omega' = \omega + m$.

Substituting (3.24) in to (3.4) produces

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{1}{8\pi}\frac{\omega'^2}{m\omega}\frac{1}{2(2m\omega)}\overline{|\mathcal{M}|^2} \\ &= \frac{1}{32\pi m^2}\left(\frac{\omega'}{\omega}\right)^2\overline{|\mathcal{M}|^2} \\ &= \frac{\pi\alpha^2}{m^2}\left(\frac{w'}{w}\right)^2\left[\frac{w'}{w} + \frac{w}{w'} - \sin^2\theta\right]. \quad (3.25) \end{aligned}$$

So, (3.19) is equal to (3.25), we have once again produced the Klein-Nishina formula.

4 Plots of differential cross section

We now have two expressions for the differential cross section. The first is with respect to t and in terms of Mandelstam variables (3.10). It may be rearranged,

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{(s-m^2)^2}\left[4m^4\left(\frac{1}{s-m^2} - \frac{1}{s-m^2+t}\right)^2 \right.$$
$$\left. + 4m^2\left(\frac{1}{s-m^2} - \frac{1}{s-m^2+t}\right) + \frac{s-m^2+t}{s-m^2} + \frac{(s-m^2)}{s-m^2+t}\right]. \quad (4.1)$$

The other is with respect to $\cos\theta$ and in terms of lab variables (3.19). It may be written,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{1}{1+\frac{w}{m}(1-\cos\theta)}\right)^2$$
$$\left[\frac{1}{1+\frac{w}{m}(1-\cos\theta)} + \frac{w}{m}(1-\cos\theta) + \cos^2\theta\right], \quad (4.2)$$

where ω is related to s by

$$\omega = \frac{s-m^2}{2m}. \quad (4.3)$$

In these forms it is simple to plot the differential cross sections for different centre of mass energies in the following limits:

1. $s - m^2 \ll m^2$ $(s > m^2)$
2. $s - m^2 \approx 2m^2$
3. $s - m^2 \gg m^2$

Note that s and t and not independent and so the limits on t are not the same in each case. Equation (3.2) relates t and ω^*, therefore, the limits on t in terms of s are

$$-\frac{(s-m^2)^2}{4s} \leq t \leq 0. \quad (4.4)$$

Note that t is always negative. Figure 4.1 shows the plots for $d\sigma/d\cos\theta$ while figure 4.2 shows those for $d\sigma/dt$.

4.1 Discussion

In the low energy limit of the lab reference frame, the graph is approximately symmetric about $\cos\theta = 0$: the likelihood of forward and backward scattering is the same. In this limit $\omega \to 0$, $\omega'/\omega \to 1$. Hence, (3.19) becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right). \quad (4.5)$$

This is the formula for Thomson scattering and may be derived within the confines of classical electromagnetism. At higher energy the probability of scattering at large angles decreases while the probability of a $\theta = 0$ scattering event is unchanged. Therefore, the total cross section decreases as s increases. In the

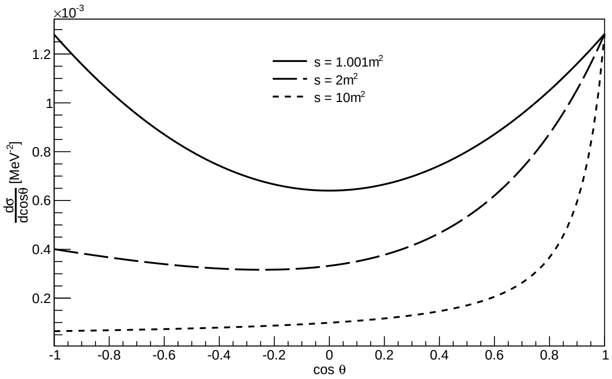
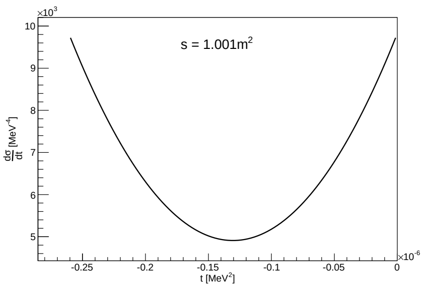
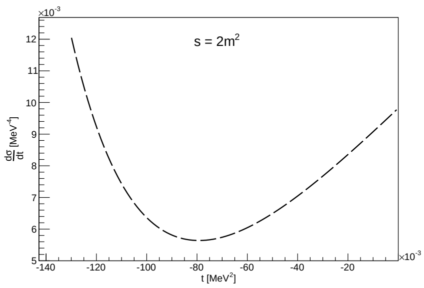


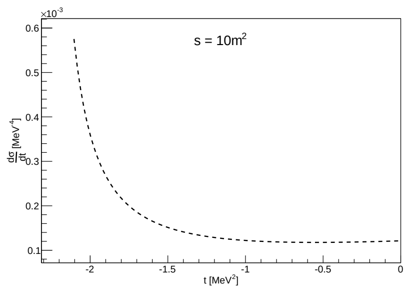
Figure 4.1: Plot of against $\cos\theta$ for three different centre of mass energies.



(a) $s - m^2 \ll m^2$



(b) $s - m^2 \approx 2m^2$



(c) $s - m^2 \gg m^2$

Figure 4.2: Plot of $d\sigma/dt$ against t for three different centre of mass energies.

high s limit the probability of backward scattering is low and fairly constant with angle, while likelihood of forward scattering increases rapidly as θ becomes smaller.

Figure 4.2 also resembles classical scattering of electromagnetic radiation for low centre of mass energy, and shows that as the centre of mass energy increases, the total cross section decreases. At high s the differential cross section is larger for higher $|t|$, deceasing rapidly as $t \to 0$. In the centre of mass this corresponds to a greater cross section for backwards scattering.

5 Conclusions

We have calculated the unpolarised differential cross section with respect to both t and $\cos\theta$. Where θ is the scattering angle in the lab frame and t is the square momentum transfer between the initial-state and final-state photon. The square matrix element $|\mathcal{M}(e^-\gamma \to e^-\gamma)|^2$ was evaluated using perturbative quantum electrodynamics at the tree level.

The Klein-Nishina formula was reproduced using two different approaches. The first by substituting $d\sigma/dt$, as calculated in the centre of mass frame, with lab frame variables by direct conversion of dt to $d\cos\theta$. The second by carrying out the calculation, including the phase space integral, directly in the lab frame.

Plots of $d\sigma/dt$ and $d\sigma/d\cos\theta$ reproduce classical scattering of electromagnetic radiation by free electrons in the low energy limit. In the lab frame, at high s, events with a small angle are more probable, peaking sharply at $\theta = 0$. The cross section decreases as the centre of mass energy increases.

A Trace evaluations for $\mathcal{A}$ and $\mathcal{A}'$

$$\begin{aligned}
\mathcal{I} &= \mathrm{tr}(\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu) \\
&= \mathrm{tr}(\not{p}'\gamma^\mu \not{k}(-2\not{p})\not{k}\gamma_\mu) \\
&= -2\mathrm{tr}(\not{p}'(-2\not{k}\not{p}\not{k})) \\
&= 4\mathrm{tr}(\not{p}'\not{k}(2p\cdot k - \not{k}\not{p})) \\
&= 32(p\cdot k)(p'\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{II} &= \mathrm{tr}(\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}2\gamma_\mu p_\nu) \\
&= 2\mathrm{tr}(\not{p}'(-2\not{p}\not{p}\not{k})p_\nu) \\
&= -4\mathrm{tr}(\not{p}'m^2\not{k}) \\
&= -16m^2(p'\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{V} &= \mathrm{tr}(\not{p}'2\gamma^\mu p^\nu \not{p}\gamma_\mu \not{k}\gamma_\mu) \\
&= 2\mathrm{tr}(\not{p}'\gamma^\mu \not{p}\not{p}\not{k}\gamma_\mu) \\
&= 2m^2\mathrm{tr}(\not{p}'(-2\not{k})) \\
&= -16m^2(p'\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{VI} &= \mathrm{tr}(\not{p}'2\gamma^\mu p^\nu \not{p}2\gamma_\mu p_\nu) \\
&= 4m^2\mathrm{tr}(\not{p}\gamma^\mu \not{p}\gamma_\mu) \\
&= -32m^2(p\cdot p')
\end{aligned}$$

$$\begin{aligned}
\mathcal{XI} &= \mathrm{tr}(m\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu) \\
&= 4m^2\mathrm{tr}(\gamma^\mu \not{k}\not{k}\gamma_\mu) = 0
\end{aligned}$$

$$\begin{aligned}
\mathcal{XII} &= \mathrm{tr}(m\gamma^\mu \not{k}\gamma^\nu \not{p}2\gamma_\mu p_\nu) \\
&= 2m^2\mathrm{tr}(\gamma^\mu \not{k}\not{p}\gamma_\mu) \\
&= 32m^2(k\cdot p)
\end{aligned}$$

$$\begin{aligned}
\mathcal{XV} &= \mathrm{tr}(m\gamma^\mu \not{k}\gamma^\nu m\gamma_\nu \not{k}\gamma_\mu) \\
&= 2m^2\mathrm{tr}(\gamma^\mu \not{p}\not{k}\gamma_\mu) \\
&= 32m^2(p\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{XVI} &= \mathrm{tr}(m2\gamma^\mu p^\nu m2\gamma_\mu p_\nu) \\
&= 4m^4\mathrm{tr}(\gamma^\mu\gamma_\mu) \\
&= 64m^4
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}' &= -tr(\not{p}\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\mu \not{k}'\gamma_\nu) \\
&= -tr(\not{p}\gamma^\mu \not{k}(-2\not{k}\gamma_\mu \not{p})) \\
&= 8tr(\not{p}'(k\cdot k')\not{p}) \\
&= 32(k\cdot k')(p\cdot p')
\end{aligned}$$

$$\begin{aligned}
\mathcal{II}' &= tr(\not{p}'\gamma^\mu \not{p}2\gamma_\nu p_\mu) \\
&= 2tr(\not{p}'\not{p}\not{k}(-2\not{p})) \\
&= 4tr(\not{p}'(2(p\cdot k) - \not{k}\not{p})\not{p}) \\
&= -32(p\cdot k)(p\cdot p') + 16m^2(p\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}' &= -tr(\not{p}'2\gamma^\mu p^\nu \not{p}\gamma_\mu \not{k}'\gamma_\nu) \\
&= -2tr(\not{p}'(-2\not{p})\not{k}'\not{p}) \\
&= 4tr(\not{p}'(2(p\cdot k') - \not{k}'\not{p})\not{p}) \\
&= 32(p\cdot k')(p\cdot p') - 16m^2(p'\cdot k')
\end{aligned}$$

$$\begin{aligned}
\mathcal{VI}' &= tr(\not{p}'2\gamma^\mu p^\nu \not{p}2\gamma_\nu p_\mu) \\
&= 4tr(\not{p}'\not{p}\not{p}\not{p}) \\
&= 16m^2(p\cdot p')
\end{aligned}$$

$$\begin{aligned}
\mathcal{XI}' &= -tr(m\gamma^\mu \not{k}\gamma^\nu m\gamma_\mu \not{k}'\gamma_\nu) \\
&= -4m^2 tr(\not{k}'\not{k}) \\
&= -16m^2(k\cdot k')
\end{aligned}$$

$$\begin{aligned}
\mathcal{XII}' &= tr(m\gamma^\mu \not{k}\gamma^\nu m2\gamma_\nu p_\mu) \\
&= 32m^2(p\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{XV}' &= -tr(m2\gamma^\mu p_\nu m\gamma_\mu \not{k}'\gamma_\nu) \\
&= 32m^2(p\cdot k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{XVI}' &= tr(m2\gamma^\mu p^\nu m2\gamma_\nu p_\mu) \\
&= 16m^4
\end{aligned}$$

References

[1] F. J. Dyson. The radiation theories of tomonaga, schwinger, and feynman. *Physical Review*, 75(3):486–502, February 1949.

[2] O. Klein. Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac. *Zeitschrift für Physik*, 53(3):157–165, March 1929.

[3] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Westview Press, Boulder, Colorado, USA, 1995.

[4] David Tong. Quantum Field Theory - University of Cambridge Part III Mathematical Tripos, 2006.