Calculation of the β function for a general SU(n) scalar QCD at one-loop level

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Abstract

We calculate the β function for a general SU(n) scalar QCD at one-loop level. The final result is

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{1}{3} C(r) \right].$$

Including QCD-like restrictions on the group we find the following result:

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[11 - \frac{1}{6} n_s \right].$$

This also demonstrates the freedom to take negative values and therefore describes a theory which allows asymptotic freedom at high energy scales.

1 Introduction

The Lagrangian for n complex scalar fields ϕ_i that are invariant under local SU(n) gauge transformations is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + \bar{\eta}^a (-\partial^\mu D^{ac}_\mu) \eta^c + (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi. \tag{1.1}$$

Scalar Quantum Chromodynamics (SQCD) arises upon quantisation of this non-abelian gauge theory and the resulting Feynman rules are given in Figure 1.1. However, if we were to use them to naively calculate the correlation function for any process involving loops it would often yield infinity. We solve this problem using the process of renormalization, a systematic way of removing infinities from physical observables such as cross sections and decay widths.

We first identify that the ‘bare’ fields and coupling constants in our Lagrangian cannot be determined experimentally, they are just parameters in our Lagrangian. If we were to perform an experiment to measure the coupling constant for a certain interaction at a certain energy scale we would be trying to measure the physical or ‘renormalized’ coupling which is directly related to the cross section for this interaction [3]. Therefore, we introduce renormalized quantities for every bare quantity in our Lagrangian. We can then consider our bare quantities to be infinite in such a way that exactly cancels the infinities arising from our loop calculations. We achieve this systematic cancellation by rewriting the Lagrangian in terms of renormalized quantities and counterterms. This procedure is outlined in Section 2. Clearly any finite term can be absorbed in to an infinite one, so there are many different renormalization schemes which define how much of the finite parts we absorb. In this report we will use the modified minimal subtraction ($\overline{\text{MS}}$) scheme, as detailed in Section 3.1.4.

In order to manipulate our divergent integrals we must first regularize them by making the integral finite. We could do this by imposing a momentum cut-off scale. In the modern view this cut-off has physical significance, it determines the scale above which new physics becomes important, for which our current model is only an effective theory. The process of renormalization ensures that when physical amplitudes are expressed in terms of renormalized parameters the cut-off dependence vanishes.

An alternative method to regulate the divergent integrals is dimensional regularization, which we will use in this report and is fundamental to the $\overline{\text{MS}}$ scheme. The procedure for this is described for the first loop calculation in Section 3.1.3. Particularly of note, this procedure introduces a mass scale μ which controls the dimensional bare coupling outside of four dimensions. The renormalized or physical coupling constant is consequently dependent on this momentum scale, i.e., the scattering amplitudes for our theory are dependent on the energy scale at which 'experiments' are performed.

In order to examine the behavior of the coupling constant $g(\mu)$ at different energy scales we define the β function,

$$\beta(g(\mu)) \equiv \frac{\partial g(\mu)}{\partial \ln(\mu)}. \tag{1.2}$$

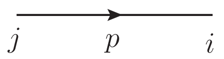
The purpose of this report is to calculate this function for SQCD. In the $\overline{\text{MS}}$ scheme this can be done by calculating three of the counterterms factors. These calculations are laid out in Sections 3, 4, and 5. The final calculation of the Beta function is performed in Section 6.

2 Renormalisation of the SQCD Lagrangian

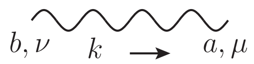
In order to distinguish renormalized, physical parameters from the unrenormalised, bare ones, we will denote the bare quantities with subscript 0 for such as m_0 and ϕ_0. We define the Lagrangian in terms of the renormalized parameters by introducing Z factors which will contain divergences,

$$\begin{aligned}
\mathcal{L} = &- \frac{1}{4}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})^2 - \bar{\eta}^a_0 \partial^\mu \eta^a_0 + \partial_\mu \phi^\dagger_0 \partial^\mu \phi_0 - m_0^2 \phi^\dagger_0 \phi_0 \\
&+ i g_0 A^{a\mu}_0 T^a [(\partial_\mu \phi_0)^\dagger \phi_0 - \phi^\dagger_0 (\partial_\mu \phi_0)] + g_0^2 \phi^\dagger_0 T^a T^b \phi_0 A^{a\mu}_0 A^b_{0\mu} \\
&- g_0 f^{abc} (\partial_\mu A^a_{0\nu}) A^{b\mu}_0 A^{c\nu}_0 + \frac{1}{4} g_0^2 (f^{eab} A^a_{0\mu} A^b_{0\nu})(f^{ecd} A^{c\mu}_0 A^{d\nu}_0) \\
&- g_0 \bar{\eta_0}^a f^{abc} \partial^\mu A^b_{0\mu} \eta^c_0 - \frac{1}{2\xi}(\partial^\mu A^a_{0\mu})^2
\end{aligned} \tag{2.1}$$

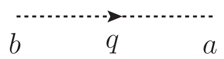
$$\begin{aligned}
\equiv &- \frac{1}{4} Z_A (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - Z_\eta \bar{\eta}^a \partial^\mu \eta^a + Z_\phi \partial_\mu \phi^\dagger \partial^\mu \phi - Z_m m^2 \phi^\dagger \phi \\
&+ i Z_1 g A^{a\mu} T^a [(\partial_\mu \phi)^\dagger \phi - \phi^\dagger (\partial_\mu \phi)] + Z_2 g^2 \phi^\dagger T^a T^b \phi A^{a\mu} A^b_\mu \\
&- Z_3 g f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} + \frac{1}{4} Z_4 g^2 (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c\mu} A^{d\nu}) \\
&- Z_5 g \bar{\eta}^a f^{abc} \partial^\mu A^b_\mu \eta^c - Z_6 \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2.
\end{aligned} \tag{2.2}$$

Scalar Propagator 

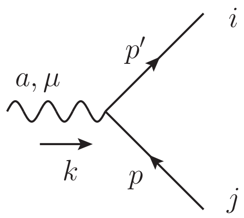
$$= \frac{i\delta_{ij}}{p^2 - m^2}$$

Vector Boson Propagator 

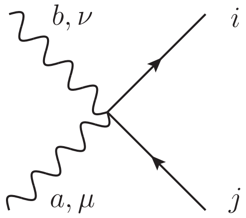
$$= \frac{-ig^{\mu\nu}\delta^{ab}}{k^2}$$

Ghost Propagator 

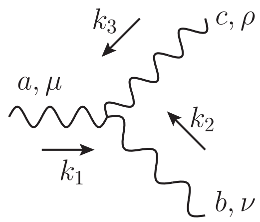
$$= \frac{-\delta^{ab}}{q^2}$$

Boson DiScalar Vertex 

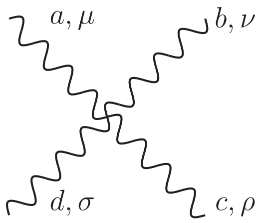
$$= ig(p + p')_\mu T^a_{ij}$$

DiBoson DiScalar Vertex 

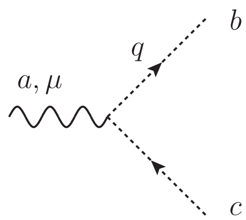
$$= ig^2 g_{\mu\nu}\{T^a, T^b\}_{ij}$$

TriBoson Vertex 

$$= gf^{abc}[(k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\rho} + (k_3 - k_1)_\nu g_{\mu\rho}]$$

QuadBoson Vertex 

$$= -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]$$

Boson DiGhost Vertex 

$$= gf^{abc}q^\mu$$

Figure 1.1: Feynman rules for SQCD. The boson propagator is shown in the Feynman gauge $\xi = 1$ [1].

From the first four terms we see that

$$\phi_0 = \sqrt{Z_\phi}\phi, \quad A_{0\mu} = \sqrt{Z_A}A_\mu, \quad \eta_0 = \sqrt{Z_\eta}\eta, \quad \text{and} \quad m_0^2 = \frac{Z_m}{Z_\phi}m^2. \tag{2.3}$$

We also see that g appears multiply in the Lagrangian. Gauge invariance suggests that all relations are equivalent

$$g = \frac{Z_\phi\sqrt{Z_A}}{Z_1}g_0 = \frac{Z_A}{\sqrt{Z_2}}g_0 = \frac{Z_A^{3/2}}{Z_3}g_0 = \frac{Z_A}{\sqrt{Z_4}}g_0 = \frac{Z_\eta\sqrt{Z_A}}{Z_5}g_0. \tag{2.4}$$

Therefore, to calculate the beta function we are required to find only the factors from one relation. We choose the relation involving Z_1, and corresponding to the Boson DiScalar vertex in Figure 1.1. This relation and vertex are selected as it has the fewest one-loop corrections.

We can now rewrite the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}, \tag{2.5}$$

where

$$\begin{aligned}\mathcal{L}_R \equiv &-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \bar{\eta}^a\partial^\mu\eta^a + \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \\ &+ igA^{a\mu}T^a[(\partial_\mu\phi)^\dagger\phi - \phi^\dagger(\partial_\mu\phi)] + g^2\phi^\dagger T^aT^b\phi A^{a\mu}A_\mu^b \\ &- gf^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} + \frac{1}{4}g^2(f^{eab}A_\mu^aA_\nu^b)(f^{ecd}A^{c\mu}A^{d\nu}) \\ &- g\bar{\eta}^af^{abc}\partial^\mu A_\mu^b\eta^c - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\mathcal{L}_{CT} \equiv &-\frac{1}{4}\delta_A(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \delta_\eta\bar{\eta}^a\partial^\mu\eta^a + \delta_\phi\partial_\mu\phi^\dagger\partial^\mu\phi - \delta_m m^2\phi^\dagger\phi \\ &+ i\delta_1 gA^{a\mu}T^a[(\partial_\mu\phi)^\dagger\phi - \phi^\dagger(\partial_\mu\phi)] + \delta_2 g^2\phi^\dagger T^aT^b\phi A^{a\mu}A_\mu^b \\ &- \delta_3 gf^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} + \frac{1}{4}\delta_4 g^2(f^{eab}A_\mu^aA_\nu^b)(f^{ecd}A^{c\mu}A^{d\nu}) \\ &- \delta_5 g\bar{\eta}^af^{abc}\partial^\mu A_\mu^b\eta^c - \delta_6\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2.\end{aligned} \tag{2.7}$$

Notice that all terms in $\mathcal{L}_{CT}$ are now included in the interacting part of the Lagrangian and hence, upon quantisation, we must derive new Feynman rules in order to compute our correlation functions.[1] The Feynman rules resulting from $\mathcal{L}_R$ are identical to those in Figure 1.1. Due to our choice of relation in (2.4), we will require just three Feynman rules for each of the three necessary counterterms. Due to the symmetry between the counterterm and bare Lagrangian these are simple to read off. We deal with the first term in (2.7) by integrating by parts and discarding the surface term to get $-\frac{1}{2}A_\mu(-\partial^2 g_{\mu\nu} + \partial^\mu\partial^\nu)A_\nu$. These new rules will be given when needed. We can now move on to calculate the three counterterms required.

[1]We will not take in to account the gauge fixing term or δ_6 in our renormalization procedure, which we effectively set to one. When fixing the gauge we will use the Feynman gauge $\xi = 1$.

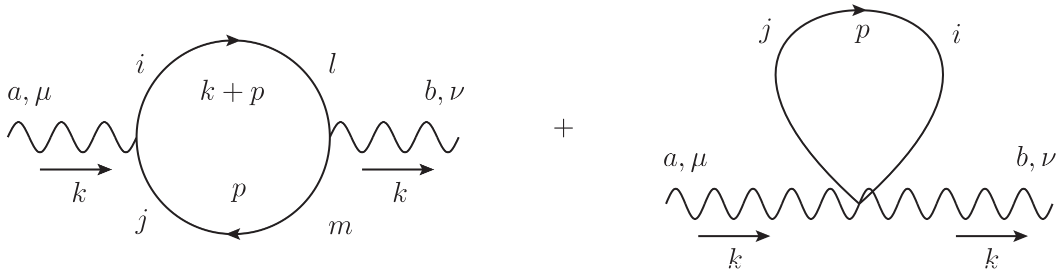


Figure 3.1: Annotated Feynman diagrams for scalar one-loop corrections to the boson propagator [1].

3 The gauge boson self-energy (vacuum polarization)

There are five fully connected, amputated one particle irreducible one-loop Feynman diagrams representing corrections to the Boson propagator. Two due to scalar loop corrections (see Figure 3.1) and 3 due to pure gauge boson self interactions. The Slavnov-Taylor identities (analogous to the ward identity in QED) imply that the boson self-energy will have a transverse Lorentz structure,

$$i\Pi_{\mu\nu}(k) = i(g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\Pi(k^2). \tag{3.1}$$

In principle we could include any number of scalar fields in our theory, it is reasonable to assume therefore that this necessary transverse structure will be found individually for the scalar loop and pure gauge sector. Consequently, we can consider these contributions separately,

$$\Pi_{\mu\nu}(k) = \Pi^s_{\mu\nu}(k) + \Pi^g_{\mu\nu}(k). \tag{3.2}$$

Comparing this to the Feynman rule due to the δ_A counterterm, we see that we can separate it,

$$-(g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\delta_A = -(g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\delta^s_A - (g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\delta^g_A. \tag{3.3}$$

We will first calculate the scalar loop self-energy corrections δ^s_A. As this is our first calculation we will describe steps in greater detail than subsequent calculations. Particularly useful results derived here will be stated in the appendices.

3.1 Scalar loop corrections to the boson propagator

Firstly, we apply the Feynman rules in Figure 1.1 to the left hand diagram inFfigure 3.1,

$$\int \frac{d^4p}{(2\pi)^4}\, ig(2p+k)_\mu T^a_{ij} \left(\frac{i\delta_{mj}}{p^2 - m^2}\right)\left(\frac{i\delta_{li}}{(p+k)^2 - m^2}\right) ig(2p+k)_\nu T^b_{ml} \tag{3.4}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, g^2\, \mathrm{tr}[T^a T^b] \frac{(2p+k)_\mu (2p+k)_\nu}{(p^2 - m^2)((p+k)^2 - m^2)}. \tag{3.5}$$

Note that for generality we have truncated the external lines such that the external line propagators do no appear in the calculation. To evaluate the trace of the group generators we must recall some basic group theory.

3.1.1 Some basic results from group theory

Our Lagrangian is invariant under a continuous SU(n) symmetry. SU(n) is the group of all $n \times n$ unitary matrices of determinant one. As an example of a simple Lie group, the generators of these transformations obey the commutation relation:

$$[T^a, T^b] = if^{abc}T^c. \tag{3.6}$$

Where f^{abc} are the antisymmetric structure constants. The matrix representations D of this group G also obey this Lie algebra. To calculate the trace over these individually traceless matrices we use the normalisation convention

$$\text{tr}[T^aT^b] = C(r)\delta^{ab}, \tag{3.7}$$

where $C(r)$ is a constant dependent on the irreducible representation r of SU(n) under which the fields transform. The generators of special unitary transformation are necessarily hermitian and so these constants are positive definite. It can also be shown that the operator

$$T^2 = T_aT_a, \tag{3.8}$$

commutes with all group generators, i.e., it is an invariant of the algebra (3.6) and as such takes a constant value on each irreducible representation,

$$T^aT^a = C_2(r)I_d, \tag{3.9}$$

where $C_2(r)$ is the quadratic Casimir operator and depends on the irreducible representation r and I_d is the $d(r) \times d(r)$ identity matrix, where d is the dimension of the r. The adjoint representation, defined by the structure constants themselves, $[T_a]_{bc} \equiv -if_{abc}$, has the same dimension as G. Therefore, we can write

$$f^{acd}f^{bcd} = C_2(G)\delta^{ab}. \tag{3.10}$$

Using (3.7) we can rewrite (3.5),

$$\int \frac{d^4p}{(2\pi)^4}\, g^2C(r)\delta^{ab}\frac{(2p+k)_\mu(2p+k)_\nu}{(p^2-m^2)((p+k)^2-m^2)}. \tag{3.11}$$

It will prove useful to calculate the two scalar loop corrections to the vacuum polarization together. So, applying the Feynman rules to the second diagram in Figure 3.1,

$$\int \frac{d^4p}{(2\pi)^4}\, ig^2g_{\mu\nu}\{T^a,T^b\}_{ji}\frac{i\delta_{ij}}{p^2-m^2} \tag{3.12}$$

$$= \int \frac{d^4p}{(2\pi)^4} - g^2g_{\mu\nu}\,\text{tr}\{T^aT^b\}\frac{((p+k)^2-m^2)}{(p^2-m^2)((p+k)^2-m^2)} \tag{3.13}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, g^2C(r)\delta^{ab}\frac{-2g_{\mu\nu}((p+k)^2-m^2)}{(p^2-m^2)((p+k)^2-m^2)}. \tag{3.14}$$

In the final equality, we again used (3.7), as well as the linearity and cyclicity of the trace to show

$$\text{tr}\{T^a,T^b\} = \text{tr}[T^aT^b] + \text{tr}[T^bT^a] = 2C(r)\delta^{ab}. \tag{3.15}$$

We then combine (3.11) and (3.14) for

$$i\Pi^s_{\mu\nu}(k) = \int \frac{d^4p}{(2\pi)^4}\, g^2 C(r)\delta^{ab} \frac{N_{\mu\nu}}{(p^2-m^2)((p+k)^2-m^2)}, \tag{3.16}$$

where

$$N_{\mu\nu} = (2p+k)_\mu (2p+k)_\nu - 2g_{\mu\nu}((p+k)^2 - m^2). \tag{3.17}$$

3.1.2 Feynman parametrization

This integral can be rewritten using the mathematical trick of Feynman parametrization. The factors in the denominator are first combined using (A.2),

$$\frac{1}{(p^2-m^2)((p+k)^2-m^2)} = \int_0^1 \frac{d\alpha}{(p^2+2\alpha p\cdot k + \alpha k^2 - m^2)^2}. \tag{3.18}$$

We then make a change of integration variable from p to $P = p + \alpha k$,

$$\int_0^1 \frac{d\alpha}{(P^2+\alpha(1-\alpha)k^2 - m^2)^2} = \int_0^1 \frac{d\alpha}{(P^2-\Delta)^2}, \tag{3.19}$$

where $\Delta = m^2 - \alpha(1-\alpha)k^2$. Shifting variables in the numerator,

$$N_{\mu\nu} =(2P + (1-\alpha)k)_\mu (2P + (1-\alpha)k)_\nu \tag{3.20}$$

$$+ 2g_{\mu\nu}(P^2 + (1-\alpha)^2 k^2 - m^2) \tag{3.21}$$

$$=4P_\mu P_\nu - 4\alpha P_\mu k_\nu - 4\alpha P_\nu k_\mu + (1-2\alpha)^2 k_\mu k_\nu + 2P_\nu k_\mu \tag{3.22}$$

$$+ 2g_{\mu\nu}(P^2 + (1-\alpha)^2 k^2 - m^2). \tag{3.23}$$

3.1.3 Dimensional regularisation

We now utilise Dimensional regulation. Firstly, we can generalize the integral to d dimensions. In order to ensure the action remains dimensionless we introduce the scale μ,

$$g^2 \int \frac{d^4P}{(2\pi)^4} \to \mu^{4-n} g^2 \int \frac{d^dP}{(2\pi)^d}. \tag{3.24}$$

This scale μ has the dimensions of mass. In introducing it we establish that the gauge coupling g remains dimensionless for all d. Effectively we shift

$$g \to g\mu^{2-\frac{d}{2}}. \tag{3.25}$$

This step is significant for the calculation of the β function. The relation for renormalized gauge coupling becomes

$$g = \mu^{\frac{d}{2}-2} \frac{Z_\phi \sqrt{Z_A}}{Z_1} g_0. \tag{3.26}$$

We can now Wick rotate from Minkowski to Euclidean spacetime. Effectively we do this by letting $P^0 = ik^0_E$, such that $d^dP = -id^dP_E$ and $P^2 = -P_E^2$.

$$i\Pi^s_{\mu\nu}(k) = ig^2 C(r)\delta^{ab} \int_0^1 d\alpha \int \frac{d^dP_E}{(2\pi)^d} \mu^{4-d} \frac{N_{\mu\nu}}{(P_E^2+\Delta)^2}. \tag{3.27}$$

We can see from (3.23) that the numerator will have some terms independent of P. In order to explore the divergent part of the integral, we will evaluate the right-most integral for a numerator independent of P (we take the numerator outside the integral),

$$i \int \frac{d^d P_E}{(2\pi)^d} \mu^{4-d} \frac{1}{P_E^2 + \Delta}. \tag{3.28}$$

This integral is of a form that often appears in loop calculations and we will use the results derived now may times. Analogously to switching to polar co-ordinates in 3D space we can write this integral as

$$= i \frac{1}{(2\pi)^d} \int \frac{d^d P_E}{(P_E^2 + \Delta)^2} = i \frac{1}{(2\pi)^d} \int \frac{d\Omega_d \, P_E^{d-1} \, dP_E}{(P_E^2 + \Delta)^2} \tag{3.29}$$

The integral over $d\Omega_d$ is the area of a surface in d dimensions,

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{3.30}$$

Generalising the power of the denominator, the following integral has solution

$$\int \frac{x^{d/2-1} dx}{(x+1)^n} = \frac{\Gamma(d/2)\Gamma(n-d/2)}{\Gamma(n)}. \tag{3.31}$$

So, in general we have

$$i \frac{1}{(2\pi)^d} \int \frac{d^d P_E}{(P_E^2 + \Delta)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}. \tag{3.32}$$

This result is repeated in (B.1) in the appendix for a calculation from Minkowski space. As the denominator is symmetric about $P_\mu = 0$ we may use symmetry arguments to simplify numerator with higher powers of P. Firstly, due to this symmetry, terms linear or cubic in P_μ vanish will vanish on integration. Also by symmetry it can be shown that

$$g^{\mu\nu} \int \frac{P_\mu P_\nu}{D} d^d P = \int \frac{P^2}{D} d^d P. \tag{3.33}$$

Hence, similarly to the calculation of (B.1), it can be shown that

$$i \frac{1}{(2\pi)^d} \int \frac{d^d P_E \, P_E^2}{(P_E^2 + \Delta)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-1}. \tag{3.34}$$

This result is also repeated in (B.2) in the appendix for a calculation from Minkowski space. Comparing (3.32) and (3.34) we see that

$$\left(\frac{2}{d} - 1\right) \int \frac{d^d P_E}{(2\pi)^d} \frac{P_E^2}{(P_E^2 + \Delta)^2} = \Delta \int \frac{d^d P_E}{(2\pi)^d} \frac{1}{(P_E^2 + \Delta)^2}. \tag{3.35}$$

Applying the above arguments we can now deal with any power of P, up to cubic terms in our integrals. We will now return to calculating the boson self energy due to scalar loops.

Using (3.33) we may effectively make the replacement $P_\mu P_\nu \to \frac{1}{d} g_{\mu\nu} P^2$ in our numerator:

$$N_{\mu\nu} = (2P + (1-\alpha)k)_\mu (2P + (1-\alpha)k)_\nu - 2g_{\mu\nu}(P^2 + (1-\alpha)^2 k^2 - m^2) \tag{3.36}$$

$$= \frac{4}{d} g_{\mu\nu} P^2 + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}(P^2 + (1-\alpha)^2 k^2 - m^2) \tag{3.37}$$

$$= 2\left(\frac{2}{d} - 1\right) g_{\mu\nu} P^2 + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}((1-\alpha)^2 k^2 - m^2) \tag{3.38}$$

$$= -2\left(\frac{2}{d} - 1\right) g_{\mu\nu} P_E^2 + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}((1-\alpha)^2 k^2 - m^2) \tag{3.39}$$

Then, due to (3.35) we can make the replacement $\left(\frac{2}{d} - 1\right) P^2 \to \Delta$ in (3.39):

$$N_{\mu\nu} = -2g_{\mu\nu}\Delta + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}((1-\alpha)^2 k^2 - m^2) \tag{3.40}$$

$$= -2g_{\mu\nu}\Delta + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}((1-\alpha)k^2 - \alpha(1-\alpha)k^2 - m^2) \tag{3.41}$$

$$= -2g_{\mu\nu}\Delta + (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}((1-\alpha)k^2 - 2\alpha(1-\alpha)k^2 - \Delta) \tag{3.42}$$

$$= (1-2\alpha)^2 k_\mu k_\nu - 2g_{\mu\nu}(1-\alpha)(1-2\alpha)k^2 \tag{3.43}$$

Now if we make a change of variable $\alpha = \beta + \frac{1}{2}$ we see that on integration terms odd in β will integrate to zero between $-\frac{1}{2}$ and $\frac{1}{2}$. This makes the numerator

$$N_{\mu\nu} = 4\beta^2 k_\mu k_\nu + 2g_{\mu\nu}\beta(1-2\beta)k^2 \tag{3.44}$$

$$= -4\beta^2(g_{\mu\nu}k^2 - k_\mu k_\nu) \tag{3.45}$$

$$= -4\beta^2 N'_{\mu\nu}, \tag{3.46}$$

where $N_{\mu\nu}$ is independent of P_E and now $\Delta = m^2 - \left(\frac{1}{4} - \beta^2\right) k^2$. Substituting this, our integral becomes

$$i\Pi^s_{\mu\nu}(k) = -4iN'_{\mu\nu} \frac{\mu^{4-d} g^2}{(2\pi)^d} C(r)\delta^{ab} \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\beta\, \beta^2 \frac{1}{(4\pi)^{d/2}} \Gamma(2 - d/2) \left(\frac{1}{\Delta}\right)^{2-d/2}, \tag{3.47}$$

where we have made use of $\Gamma(2) = (2-1)! = 1$.

3.1.4 Applying the $\overline{\text{MS}}$ scheme

We are now close to isolating the divergent part of our integral. We now define a quantity ϵ by $d = 4 - 2\epsilon$. Notice that at $d = 4$, $\epsilon = 0$ and (3.47) is undefined because $\Gamma(0) = (-1)! = \bar{\infty}$. Therefore, we perform a Laurent expansion about $d = 4$,

$$i\frac{\Gamma(\epsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + \ln\left(\frac{4\pi\mu^2}{\Delta}\right) - \gamma_E + \mathcal{O}(\epsilon)\right) \tag{3.48}$$

$$= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{\Delta}{\mu^2}\right) + \mathcal{O}(\epsilon)\right). \tag{3.49}$$

Neglecting terms which vanish as $\epsilon \to 0$,

$$i\frac{\Gamma(\epsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{\Delta}{\mu^2}\right)\right) \tag{3.50}$$

$$= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}\right) + \text{finite}. \tag{3.51}$$

This collection of terms will always arise: the divergent part as $\epsilon \to 0$ and a collection of physically meaningless constants. In the $\overline{\text{MS}}$ scheme, we absorb the divergent part and a universal constant into the counterterms. The finite parts are included in our renormalized 1PI Green's functions. In future, as we are only concerned with the divergent part, we will frequently suppress any terms that only contribute to the finite part of the solution.

Returning to the self-energy calculation and using the results above,

$$i\Pi^s_{\mu\nu}(k) = \frac{-4g^2C(r)}{(4\pi)^2} iN'_{\mu\nu}\delta^{ab} \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\beta\,\beta^2 \left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m^2 - (1/4-\beta^2)k^2}{\mu^2}\right)\right] \tag{3.52}$$

$$= \frac{-4g^2C(r)}{(4\pi)^2} iN'_{\mu\nu}\delta^{ab} \left[\frac{1}{12}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right) - \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\beta\,\beta^2 \ln\left(\frac{m^2 - (1/4-\beta^2)k^2}{\mu^2}\right)\right] \tag{3.53}$$

$$= i\Pi^s(k^2)(k^2 g_{\mu\nu} - k_\mu k_\nu)\delta^{ab}, \tag{3.54}$$

where,

$$\Pi^s(k^2) = -\frac{1}{3}\frac{g^2C(r)}{16\pi^2}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right) + \text{finite}. \tag{3.55}$$

It is now time to include the relevant Feynman rule due to the Counterterm Lagrangian. If we define $i\Pi^s_{R\mu\nu}(k)$ to be the finite self-energy correction, then we have the relation:

$$i\Pi^s_{R\mu\nu}(k) = i\Pi^s_{\mu\nu}(k) - (g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\delta^s_A. \tag{3.56}$$

Therefore, we identify the counterterm to be

$$\delta^s_A = -\frac{1}{3}\frac{g^2C(r)}{16\pi^2}\left(\frac{1}{\epsilon}\right). \tag{3.57}$$

3.2 Total counterterm factor for the boson propagator

The 3 self-energy corrections due purely to the gauge boson self interactions are identical to those in regular spinor QCD. For a detailed calculation of these diagrams see [2]. The total counterterm factor for the pure gauge sector corrections is

$$\delta^g_A = \frac{5}{3}\frac{g^2C_2(G)}{16\pi^2}\frac{1}{\epsilon}. \tag{3.58}$$

For the total counterterm factor for the Boson Propagator, we simply sum the contributions in (3.57) and (3.58)

$$\delta_A = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3}C_2(G) - \frac{1}{3}C(r)\right]\frac{1}{\epsilon}. \tag{3.59}$$

4 Scalar self-energy

There are two fully connected, amputated one particle irreducible one-loop Feynman diagrams corrections to the Scalar propagator. These are shown in Figure 4.1 in addition to the diagram due to the counterterm Lagrangian. Adding the contribution due to the final diagram introduces

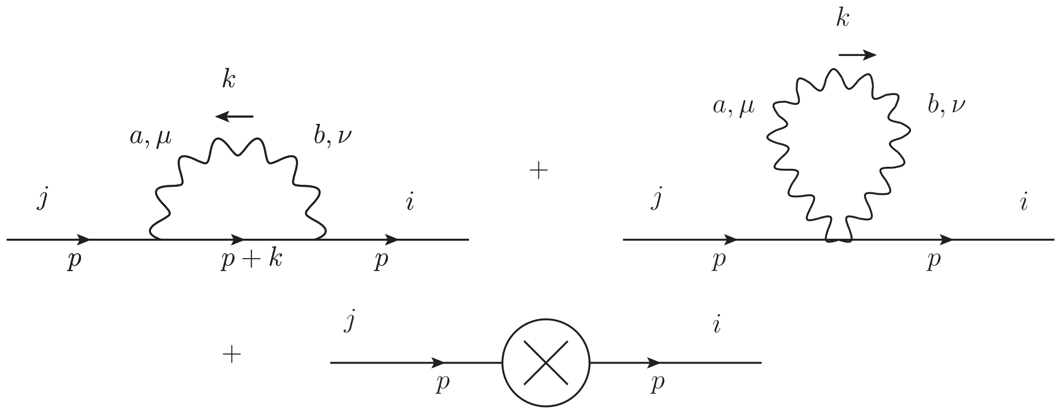


Figure 4.1: Annotated Feynman diagrams for one-loop and counterterm corrections to the scalar propagator [1].

an additional term $+i(p^2\delta_\phi - m^2\delta_m)\delta_{ij}$. Including this will give us the finite scalar self-energy $i\Sigma^R_{ij}(p^2, m)$. Applying Feynman rules to the first loop diagram in Figure 4.1,

$$
\begin{aligned}
&\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\, ig(k+2p)_\mu T^a_{mj}\frac{-i\delta^{ab}g^{\mu\nu}}{k^2-\lambda^2}\frac{i\delta_{lm}}{(p+k)^2-m^2} ig(k+2p)_\nu T^b_{il} \\
=&-g^2[T^aT^a]_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\,\frac{(k+2p)^2}{(k^2-\lambda^2)((k+p)^2-m^2)} \\
=&-g^2C_2(r)\delta_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\,\frac{(k+2p)^2}{(k^2-\lambda^2)((k+p)^2-m^2)}.
\end{aligned}
\tag{4.1}
$$

In the final equality we used (3.9). Applying Feynman rules to the second loop diagram in Figure 4.1,

$$
\begin{aligned}
&\frac{1}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\, ig^2 g_{\mu\nu}\{T^a,T^b\}_{ij}\frac{-i\delta^{ab}g^{\mu\nu}}{k^2-\lambda^2} \\
=&\frac{1}{2}g^2 g^{\mu\nu}g_{\mu\nu}\{T^aT^a\}_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\,\frac{(k+p)^2-m^2}{(k^2-\lambda^2)((k+p)^2-m^2)} \\
=&g^2dC_2(r)\delta_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\,\frac{(k+p)^2-m^2}{(k^2-\lambda^2)((k+p)^2-m^2)}.
\end{aligned}
\tag{4.2}
$$

Where we've again used cyclicity of the trace. To calculate the full correction due to the loop diagrams, including divergences, we combine (4.1) and (4.2),

$$
i\Sigma_{ij}(p^2, m) = -g^2C_2(r)\delta_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\,\frac{N}{(k^2-\lambda^2)((k+p)^2-m^2)}. \tag{4.3}
$$

Where the numerator is

$$
N = (k+2p)^2 - d(k+p)^2 + dm^2. \tag{4.4}
$$

As before we can introduce a Feynman parameter using (A.2),

$$\frac{1}{(k^2-\lambda^2)((k+p)^2-m^2)} = \int_0^1 d\alpha \, \frac{1}{(k^2-(1-\alpha)\lambda^2+2\alpha p\cdot k+\alpha p^2-\alpha m^2)^2} \tag{4.5}$$

$$= \int_0^1 d\alpha \, \frac{1}{(K^2-\Delta)^2}. \tag{4.6}$$

Where $K = k+\alpha p$ and $\Delta = \alpha m^2 - \alpha(1-\alpha)p^2 + (1-\alpha)\lambda^2$. This makes the numerator:

$$N = (K+(2-\alpha)p)^2 - d(K+(1-\alpha)p)^2 + dm^2 \tag{4.7}$$

$$= K^2 + (2-\alpha)^2 p^2 - dK^2 - d(1-\alpha)^2 p^2 + dm^2 + \mathcal{O}(K) \tag{4.8}$$

$$= (1-d)K^2 + [(2-\alpha)^2 - d(1-\alpha)^2]p^2 + dm^2. \tag{4.9}$$

We have again discarded terms linear in K as they will integrate symmetrically to nought. Our integral is now of the form:

$$i\Sigma_\phi(p^2,m) = -g^2 C_2(r)\delta_{ij} \int \mu^{4-d} \frac{d^d k}{(2\pi)^d} \int_0^1 d\alpha \, \frac{Ak^2+B}{(K^2-\Delta)^2}. \tag{4.10}$$

Where $A = 1-d$ and $B = [(2-\alpha)^2 - d(1-\alpha)^2]p^2 + dm^2$. Using (3.35), we can make the replacement $Ak^2 \to (1-d/2)^{-1}A\Delta$. So, the integral becomes

$$i\Sigma_\phi(p^2,m) = -g^2 C_2(r)\delta_{ij} \int_0^1 d\alpha \left[\frac{1}{1-d/2}A\Delta + B\right] \int \mu^{4-d} \frac{d^d k}{(2\pi)^d} \frac{1}{(K^2-\Delta)^2}. \tag{4.11}$$

Using (B.4), the rightmost integral will ultimately be evaluated to

$$i\Sigma_\phi(p^2,m) = \frac{-ig^2}{(4\pi)^2} C_2(r)\delta_{ij} \int_0^1 d\alpha \left[\frac{1}{1-d/2}A\Delta + B\right] \left(\frac{1}{\epsilon} + \text{finite}\right). \tag{4.12}$$

So, we must simply evaluate the integral over $d\alpha$ inside the square brackets which is

$$\int_0^1 d\alpha \left[\frac{1-d}{1-d/2}(\alpha m^2 - \alpha(1-\alpha)p^2 + (1-\alpha)\lambda^2) + ((2-\alpha)^2 - d(1-\alpha)^2)p^2 + dm^2\right]. \tag{4.13}$$

Since we have isolated the terms divergent as $d \to 4$ we may now work in four dimensions,

$$= \int_0^1 d\alpha \left[-6(\alpha m^2 - \alpha(1-\alpha)p^2 + (1-\alpha)\lambda^2) + ((2-\alpha)^2 - 4(1-\alpha)^2)p^2 + 4m^2\right] \tag{4.14}$$

$$= \left(m^2 + 2p^2 + \frac{1}{2}\lambda^2\right). \tag{4.15}$$

The artificial photon mass may be set to zero, so our final solution is

$$i\Sigma_\phi(p^2,m) = -ig^2 C_2(r)\delta_{ij}\,(m^2+2p^2)\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{4.16}$$

Reintroducing the Feynman diagram due to the counterterm Lagrangian,

$$i\Sigma^R_{ij}(p^2,m) = -ig^2 C_2(r)\delta_{ij}\,(m^2+2p^2)\left(\frac{1}{\epsilon} + \text{finite}\right) + i(p^2\delta_\phi - m^2\delta_m)\delta_{ij}. \tag{4.17}$$

From this we can identify that, in order to cancel the divergent parts, the counterterm factor must be

$$\delta_\phi = 2\frac{g^2}{(4\pi)^2} C_2(r)\frac{1}{\epsilon}. \tag{4.18}$$

5 Corrections to the gluon-discalar vertex

There are five fully connected, amputated one particle irreducible one-loop Feynman diagrams representing corrections to Boson-DiScalar vertex, these are show in Figure 5.1. Adding all these loop corrections together will produce the Vertex correction $i\Lambda_{\mu ij}(p,p') = i(p+p')T^a_{ij}\Lambda$. We will find that diagrams 1 and 2 recover this structure together while diagrams 3 and 4 recover it separately. The fifth will not contribute. Therefore, we are free to label these contributions by $i\Lambda^M_{\mu ij}(p,p')$ where $M = 1,3,4$. Adding the contribution due to the sixth diagram in Figure 5.1 introduces an additional term $i\delta_1 g(p+p')_\mu T^a_{ij}$ which will give us the finite Vertex correction $i\Lambda^R$.

5.1 First and second vertices

Applying the Feynman rules to diagram one in Figure 5.1,

$$ig^2 g_{\mu\nu}\left(\frac{-ig^{\nu\sigma}\delta^{bc}}{k^2-\lambda^2}\right)\left(\frac{i\delta_{nl}}{(p+k)^2-m^2}\right)ig(k+2p')_\sigma T^c_{in} \tag{5.1}$$

$$= -g^3(T^b\{T^a,T^b\})_{ij}\int \mu^{4-d}\frac{d^dk}{(2\pi)^d}\frac{(k+2p')_\mu}{(k^2-\lambda^2)((p'+k)^2-m^2)}. \tag{5.2}$$

We can simply the group generators using (3.9) and (3.10) using

$$\begin{aligned}T^b\{T^a,T^b\} =& T^bT^aT^b + T^b[T^a,T^b]\\ =& C_2(r)T^a + if^{abc}T^bT^c\\ =& C_2(r)T^a + \frac{1}{2}if^{abc}(T^bT^c - T^cT^b)\\ =& C_2(r)T^a - \frac{1}{2}f^{abc}f^{bcd}T^d\\ =& [C_2(r)T^a - \frac{1}{2}C_2(G)T^a].\end{aligned} \tag{5.3}$$

Note that the denominator is identical to that in (4.1) so follows the same Feynman parametrization and necessitates the same variable shift to $K = k+\alpha p'$. We can then disregard the K linear term as always. Therefore, we have

$$-g^3p'_\mu T^a_{ij}[2C_2(r) - \frac{1}{2}C_2(G)]\int_0^1 d\alpha(2-\alpha)\int \mu^{4-d}\frac{d^dK}{(2\pi)^d}\frac{1}{(K^2-\Delta)^2}. \tag{5.4}$$

The integral over α is trivially 3/2, while the right hand integral is once again solved using (B.2) and (B.4). Consequently, we can instantly write

$$-\frac{3}{2}\frac{g^3}{16\pi^2}p'_\mu T^a_{ij}[2C_2(r) - \frac{1}{2}C_2(G)]\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{5.5}$$

To evaluate the next diagram we note that the second diagram in Figure 5.1 represents an identical matrix element up to an interchanging of p' with p. Combining these similar contributions yields

$$i\Lambda^1_{\mu ij}(p,p') = -ig(p+p')_\mu T^a_{ij}\frac{g^2}{16\pi^2}[3C_2(r) - \frac{3}{4}C_2(G)]\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{5.6}$$

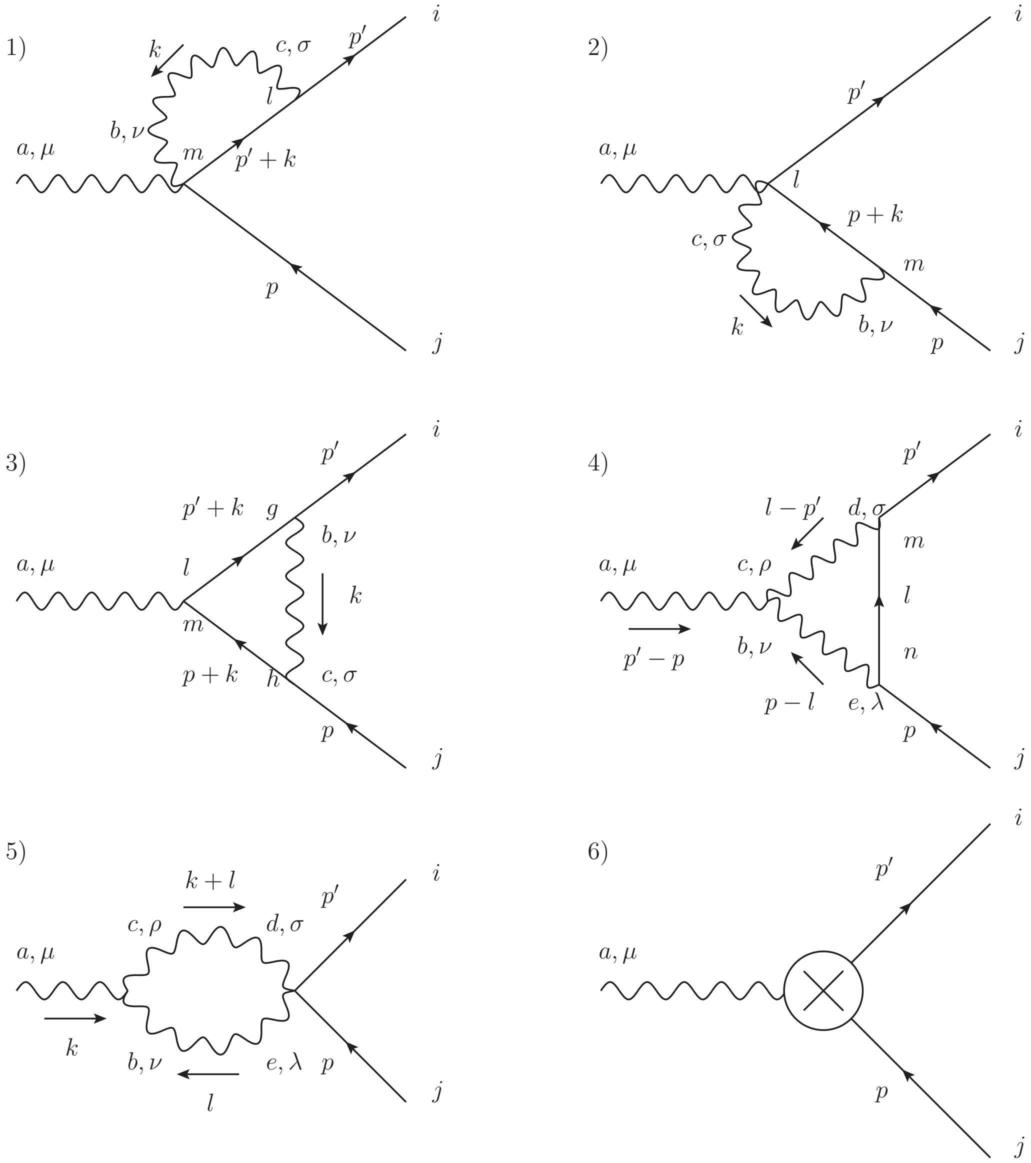


Figure 5.1: Annotated Feynman diagrams for one-loop and counterterm corrections to the gluon-discalar vertex [1].

5.2 Third vertex

Applying the Feynman rules to the third diagram in Figure 5.1,

$$\begin{aligned}
i\Lambda^3_{\mu ij}(p,p') =& \mu^{4-d}\int \frac{d^dK}{(2\pi)^d} ig(p+p')_\mu T^a_{lm}\left(\frac{i\delta_{mh}}{(p+K)^2-m^2}\right)\left(\frac{i\delta_{gl}}{(p'+k)^2-m^2}\right) \\
&\times ig(2p'+k)_\nu T^b_{ig}\left(\frac{-ig^{\sigma\nu}\delta^{bc}}{k^2-\lambda^2}\right) ig(2p+k)_\sigma T^c_{hj} \\
=& g^3[C_2(r)-\frac{1}{2}C_2(G)]T^a_{ij}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{N_\mu(k)}{(k^2-\lambda^2)((p+k)^2-m^2)((p'+k)^2-m^2)}
\end{aligned} \tag{5.7}$$

Where we've again used (5.3). We can deal with this denominator with Feynman parametrization (A.3),

$$\begin{aligned}
&\frac{1}{(k^2-\lambda^2)((p+k)^2-m^2)((p'+k)^2-m^2)} = \\
&\qquad 2\int_0^1\int_0^1 \frac{d\alpha d\beta}{[k^2-\lambda^2+(p^2+2k\cdot p-m^2+\lambda^2)\alpha+(p'^2+2k\cdot p'-m^2+\lambda^2)\beta]^3}.
\end{aligned} \tag{5.8}$$

We shift integration variables to $K = k + p\alpha + p'\beta$. This makes our denominator

$$= 2\int_0^1\int_0^1 \frac{d\alpha d\beta}{(K^2-\Delta)^3}, \tag{5.9}$$

where $\Delta = m^2(\alpha+\beta)+\lambda^2(1-\alpha-\beta)+2\alpha\beta p\cdot p'$, and our numerator

$$N_\mu(K) = (2K+(1-2\alpha)p+(1-2\beta)p')_\mu(K+(2-\beta)p'-\alpha p)^\sigma(K+(2-\alpha)-\beta p')_\sigma. \tag{5.10}$$

So, we have

$$i\Lambda^3_{\mu ij}(p,p') = 2g^3[C_2(r)-\frac{1}{2}C_2(G)]T^a_{ij}\int_0^1\int_0^1 d\alpha d\beta\int \mu^{4-d}\frac{d^dK}{(2\pi)^d}\frac{N_\mu(K)}{(K^2-\Delta)^3}. \tag{5.11}$$

From (5.10) we can see that once expanded the numerator will have general form $N_\mu(K) = A_\mu K^2 + B_\mu K + C_\mu K^0$. As before, terms linear in K will integrate to nought, so we can neglect B_μ. We can also see that terms independent of K will integrate like

$$\int \frac{d^dK}{(2\pi)^d}\frac{C_\mu}{(K^2-\Delta)^3} = \frac{(-1)^3C_\mu}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-d/2}. \tag{5.12}$$

Instantly we can see that with $d = 4$ this equation is convergent and therefore will not contribute to the divergent part of the solution. Expanding (5.10) and keeping only quadratic terms we get

$$A_\mu K^2 = [(1-2\alpha)p_\mu+(1-2\beta)p'_\mu]K^2 - 4K_\mu K_\nu[(1-\alpha)p^\nu+(1-\beta)p^\nu] \tag{5.13}$$

$$= [(1-2\alpha-4/d+4\alpha/d)p_\mu+(1-2\beta-4/d+4\beta/d)p'_\mu]K^2. \tag{5.14}$$

So our integral becomes

$$i\Lambda^3_{\mu ij}(p,p') = 2g^3[C_2(r)-\frac{1}{2}C_2(G)]T^a_{ij}\int_0^1\int_0^1 d\alpha d\beta\, A_\mu\int \mu^{4-d}\frac{d^dK}{(2\pi)^d}\frac{K^2}{(K^2-\Delta)^3} \tag{5.15}$$

$$= 2i\frac{g^3}{(4\pi)^4}[C_2(r)-\frac{1}{2}C_2(G)]T^a_{ij}\int_0^1\int_0^1 d\alpha d\beta\, A_\mu\left(\frac{1}{\epsilon}+\text{finite}\right). \tag{5.16}$$

Restoring $d = 4$ in (5.14) and isolating the integration of the numerator A_μ over the Feynman parameters,

$$\int_0^1 \int_0^1 d\alpha d\beta(-\alpha p - \beta p')_\mu = -\frac{1}{2}(p + p')_\mu. \tag{5.17}$$

So, the final solution for this diagram is

$$i\Lambda^3_{\mu ij}(p, p') = -ig(p + p')_\mu T^a_{ij} \frac{g^2}{(4\pi)^2} \left[C_2(r) - \frac{1}{2}C_2(G)\right] \left(\frac{1}{\epsilon} + \text{finite}\right). \tag{5.18}$$

5.3 Fourth vertex

Applying the Feynman rules,

$$\begin{aligned} i\Lambda^4_{\mu ij}(p, p') =& \frac{1}{2}\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} g f^{abc}[(p' - 2p + l)_\rho g_{\mu\nu} + (p - 2l + p')_\mu g_{\nu\rho} + (l - 2p' + p)_\nu g_{\mu\rho}] \\ &\times \left(\frac{-ig^{\sigma\rho}\delta^{dc}}{(l - p')^2 - \lambda^2}\right) \left(\frac{-ig^{\lambda\nu}\delta^{eb}}{(p - l)^2 - \lambda^2}\right) \left(ig(p' + l)_\sigma T^d_{im}\right) \left(ig(p + l)_\lambda T^e_{nj}\right) \left(\frac{i\delta^{mn}}{l^2 - m^2}\right) \\ =& \frac{i}{2} g^3 f^{abc}[T^c T^b]_{ij} \int \frac{d^d l}{(2\pi)^d} \mu^{4-d} \frac{N_\mu}{(l^2 - m^2)((p - l)^2 - \lambda^2)((l - p')^2 - \lambda^2)}. \end{aligned} \tag{5.19}$$

We can simply the products of generators using one of the steps in (5.3). The numerator is given by

$$N_\mu = [(p' - 2p + l)_\rho g_{\mu\nu} + (p - 2l + p')_\mu g_{\nu\rho} + (l - 2p' + p)_\nu g_{\mu\rho}](p' + l)_\rho (p + l)^\nu \tag{5.20}$$

$$\begin{aligned} =& [(k + p' - 2p)_\rho (p' + k)\rho(p + k)_\mu \\ &+ (-2k + p' + p)_\mu (p' + k)_\nu (p + k)^\nu \\ &+ (k - 2p' + p)_\nu (p + k)^\nu (p' + k)_\mu]. \end{aligned} \tag{5.21}$$

Note that the denominator in (5.19) is identical to that in (5.7) and so we perform the same Feynman parametrization and shift of variable. Also, similarly to the previous diagram, we know that, due to the form of the denominator, only terms quadratic in l will contribute to the counterterm. Hence, the relevant part of our numerator is

$$\begin{aligned} A_\mu K^2 =& 2K^2(p + p')_\mu - 2K_\mu K_\nu (p + p')^\nu \\ =& 2K^2(p + p')_\mu - 2\frac{K^2}{d}(p + p')_\mu \end{aligned} \tag{5.22}$$

$$\begin{aligned} A_\mu =& 2(p + p')_\mu (1 - 1/d) \\ =& \frac{3}{2}(p + p')_\mu. \end{aligned} \tag{5.23}$$

A_μ is independent of α and β, so the integral over the Feynman parameters was performed trivially. Having isolated the divergent part we returned to $d = 4$ dimensions. Between (5.22) and (5.23) we perform the integral over K^2 identically to that in (5.16). Assembling the above, the solution for this diagram is

$$i\Lambda^4_{\mu ij}(p, p') = -ig(p + p')_\mu T^a_{ij} \frac{g^2}{(4\pi)^2} \left[\frac{3}{4}C_2(G)\right] \left(\frac{1}{\epsilon} + \text{finite}\right). \tag{5.24}$$

5.4 Fifth vertex

Applying the Feynman diagrams to the fifth Vertex:

$$
\begin{aligned}
i\Lambda^5_{\mu ij}(p,p') =& \frac{1}{2}\mu^{4-d}\int \frac{d^d l}{(2\pi)^d} g f^{abc}\left[(k-l)g_{\mu\nu} + (2l+k)_\mu g_{\nu\rho} + (-2k-l)_\nu g_{\mu\rho}\right] \\
& \times \left(\frac{-ig^{\sigma\rho}\delta^{dc}}{(k+l)^2-\lambda^2}\right)\left(\frac{-ig^{\nu\lambda}\delta^{be}}{l^2-\lambda^2}\right) ig^2 g_{\lambda\sigma}\{T^eT^d\}_{ij} \qquad (5.25)\\
=& \frac{i}{2}g^3\left[f^{abc}\{T^b,T^c\}_{ij}\right]\mu^{4-d}\int \frac{d^d l}{(2\pi)^d}\frac{N_\mu}{((k+l)^2-\lambda^2)(l^2-\lambda^2)}. \qquad (5.26)
\end{aligned}
$$

Using the antisymmetry of the structure constants, a quick analysis of the factor in square parentheses in (5.26) shows that this diagram does not contribute.

$$
f^{abc}(T^bT^c + T^cT^b) = f^{abc}(T^bT^c - T^bT^c) = 0. \qquad (5.27)
$$

5.5 Total vertex correction

We may now sum all the one-loop corrections to the boson discalar vertex (5.6), (5.18) and (5.24). The result is

$$
i\Lambda_{\mu ij}(p,p') = -ig(p+p')_\mu T^a_{ij}\frac{g^2}{(4\pi)^2}\left[-2C_2(r) + C_2(G)\right]\left(\frac{1}{\epsilon} + \text{finite}\right). \qquad (5.28)
$$

Our finite amplitude is given by including the contribution due to Diagram 6 in Figure 5.1,

$$
i\Lambda^R_{\mu ij}(p,p') = i\Lambda_{\mu ij}(p,p') + i\delta_1 g(p+p')_\mu T^a_{ij}. \qquad (5.29)
$$

Therefore, the total counterterm corresponding to this vertex must be

$$
\delta_1 = \frac{g^2}{(4\pi)^2}\left[2C_2(r) - C_2(G)\right]\frac{1}{\epsilon}. \qquad (5.30)
$$

6 Calculating the β function

Having calculated all of the necessary counterterms we may calculate the Beta function. From (3.26) we have

$$
\begin{aligned}
g =& \mu^{\frac{d}{2}-2}\frac{Z_\phi\sqrt{Z_A}}{Z_1}g_0 \qquad (6.1)\\
=& \mu^{-\epsilon}Z_g g_0, \qquad (6.2)
\end{aligned}
$$

where,

$$
\begin{aligned}
Z_g =& Z_\phi Z_A^{\frac{1}{2}} Z_1^{-1} \qquad (6.3)\\
=& (1+\delta_\phi)(1+\delta_A)^{\frac{1}{2}}(1+\delta_1)^{-1} \qquad (6.4)\\
=& (1+\delta_\phi)\left(1 + \frac{1}{2}\delta_A - \frac{1}{8}\delta_A^2 + \cdots\right)\left(1 - \delta_1 + \delta_1^2 + \cdots\right) \qquad (6.5)
\end{aligned}
$$

As we have found, the $\delta's$ are proportional to g^2, so we can multiply this out to get:

$$Z_g = 1 + \delta_\phi - \delta_1 + \frac{1}{2}\delta_A + \mathcal{O}(g^4). \tag{6.6}$$

Substituting the counterterm factors from (3.59), (4.18) and (5.30) we get:

$$Z_g - 1 = \frac{g^2}{(4\pi)^2}\left[2C_2(r) - 2C_2(r) + C_2(G) + \frac{5}{6}C_2(G) - \frac{1}{6}C(r)\right]\frac{1}{\epsilon} \tag{6.7}$$

$$= \frac{g^2}{(4\pi)^2}\left[\frac{11}{6}C_2(G) - \frac{1}{6}C(r)\right]\frac{1}{\epsilon}, \tag{6.8}$$

which has the basic form:

$$Z_g = 1 + g^2 a\frac{1}{\epsilon}, \tag{6.9}$$

where,

$$a = \frac{1}{(4\pi)^2}\left[\frac{11}{6}C_2(G) - \frac{1}{6}C(r)\right]. \tag{6.10}$$

Now that we are working in a general d dimensions, we must define our β function in d dimensions as,

$$\tilde{\beta}(g) = \frac{\partial g(\mu)}{\partial \ln(\mu)} \tag{6.11}$$

Substituting (6.2) gives,

$$\tilde{\beta} = -\epsilon g + \frac{g}{Z_g(g)}\frac{\partial Z_g(g)}{\partial g}\tilde{\beta}. \tag{6.12}$$

$\tilde{\beta}$ is finite as $\epsilon \to 0$ so comparing the coefficient of ϵ we must have

$$\tilde{\beta} = -\epsilon g + \beta. \tag{6.13}$$

Substituting this in to the right hand side of the previous equation gives us,

$$Z_g(g)\beta = -\epsilon g^2\frac{\partial Z_g(g)}{\partial g} + \beta\frac{\partial Z_g(g)}{\partial g}g. \tag{6.14}$$

Substituting (6.9) and comparing coefficients of ϵ means that the β function in 4 dimensions is given by

$$\beta = -2g^2 ag \tag{6.15}$$

$$= -2g^2\frac{1}{(4\pi)^2}\left[\frac{11}{6}C_2(G) - \frac{1}{6}C(r)\right]. \tag{6.16}$$

Therefore, our final solution for the β function of scalar QCD is

$$\beta = -g^3\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{1}{3}C(r)\right]. \tag{6.17}$$

7 Conclusion

We have calculated the β function of Scalar Quantum Chromodynamics with a general SU(n) symmetry at one-loop level in the minimal subtraction scheme. The result was

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{1}{3} C(r) \right]. \tag{7.1}$$

We can compare this to the well know result for the same theory with scalars substituted by spinors, which is given by

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]. \tag{7.2}$$

Where, n_f is the number of fermions. Note that we could have added any number of scalar fields n_s we wished to our scalar theory for a similar factor. In standard QCD, we know that gauge fields transform in the adjoint representation of SU(3). So $C_2(G) = N_c = 3$, where N_c is the number of colours. Also, the fermions in QCD transform under the fundamental representation, hence $C(r) = \frac{1}{2}$. This means that the β function for regular QCD is

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[11 - \frac{2}{3} n_f \right]. \tag{7.3}$$

If we make similar restrictions on (7.1), we get

$$\beta = -g^3 \frac{1}{(4\pi)^2} \left[11 - \frac{1}{6} n_s \right]. \tag{7.4}$$

Note that the general form of the two β functions are very similar. In particular both can take negative values. This means that the physical coupling will decrease with increasingly energy scale leading to the famous phenomenon known as asymptotic freedom. Consequently both 'scalar quarks' and spinor quarks interact weakly at high energies. Therefore, we can be confident that asymptotic freedom is a persistent consequence of non-abelian gauge symmetries and the corresponding Yang-Mills Lagrangian, independent of the spins of the interacting matter.

References

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A Feynman parametrization

The general formula for Feynman parametrization is

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 d\alpha_2 \cdots d\alpha_n}{(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)^n} \delta \left(1 - \sum_{i=1}^{n} \alpha_i \right). \tag{A.1}$$

Specific cases reduce to

$$\frac{1}{ab} = \int_0^1 \frac{d\alpha}{\left[a + (b-a)\alpha\right]^2} \tag{A.2}$$

and

$$\frac{1}{abc} = 2\int_0^1 \int_0^1 \frac{d\alpha d\beta}{\left[a + (b-a)\alpha + (c-a)\beta\right]^3}. \tag{A.3}$$

B Dimensional regularization equations

Summarising Section 3.1.3, the following equations are useful throughout this report.

$$\frac{1}{(2\pi)^d} \int \frac{d^d P}{\left(P^2 - \Delta\right)^n} = i \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}. \tag{B.1}$$

$$i \frac{1}{(2\pi)^d} \int \frac{d^d P \, P^2}{\left(P^2 - \Delta\right)^n} = i \frac{(-1)^{n-1}}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-1}. \tag{B.2}$$

$$i \frac{\Gamma(\epsilon)}{(4\pi)^2} \left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon} = \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{\Delta}{\mu^2}\right)\right) \tag{B.3}$$

$$= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon}\right) + \text{finite}. \tag{B.4}$$